COMPUTER AIDED DESIGN OF FIR DIGITAL FILTER

By

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AUGUST 1985.

To

my parents.

DECLARATION

I do hereby declare that neither this thesis nor any part thereof has been submitted or is being concurrently submitted in candidature for any degree at any other university.

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CIRTIFICATE OF RESEARCH

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ABSTRACT

The use of optimization technique for designing digital filters is of widespread practice. For a given order, filters designed by using the iterative methods can closely approximates the 'best' filter for that order than by using other methods.

In the present work, a design technique using Hooke and Jeeves pattern search technique has been implemented. The design procedure is based on frequency sampling method. Optimization technique has also been implemented to define the transition region of the desired filter to maximise the stop band attenuation.

An algorithm has also been presented which gives the solution for designing a filter, when the pass band and stop band frequencies and maximum tolerable deviations in pass band and stop band are given but the order of the filter is not given.

The developed design technique has been implemented on a IBM 4331 Computer using FORTRAN language.

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Contd...

CHAPTER I

INTRODUCTION

1. INTRODUCTION

1.1 GENERAL

The digital filter is a digital system that can be used to filter discrete-time signals[10]. It is the digital signal processor that converts a sequence of numbers called the input to another sequence of numbers called the output. A discrete-time signal, like a continuous-time signal[3], can be represented by a unique function of frequency referred to as the frequency spectrum of the signal. This is a description of the frequency content of the signal. Filtering is the particular process by which the frequency spectrum of a signal can be modified, reshaped, or manipulated according to some desired specification. It can be implemented by means of soft-ware or by means of dedicated hard-ware, and in either case it can be used to filter real-time signals or non-real-time (recorded) signals.

The advantages of digital filters over analog filters are the traditional advantages associated with digital systems in general:

(1) Accuracy is high

(2) Physical size is small

(3) Reliability is high

A very important additional advantage of digital filters is the ease with which filter parameters can be changed in order to change the filter characteristics.

As digital computers came into scene, digital filters began to proliferate. Digital filters have been used by the scientists to solve many interesting problems, some of its uses are like in picture processing which uses digital filtering techniques to improve the clarity of pictures obtained from remote sensing, interplanetory communications, x-ray films. Other area of application include speech processing, radar, sonar, and various fields of medical technology.

1.2 TERMINOLOGY

Before discussing the design issues, it is important to distinguish the various types of digital filters.

1) Finite-Impulse-Response (FIR): This term means that the duration of the filter impulse response h(n) is finite, i.e.

$$h(n) = 0, \quad n > N_1 < \infty$$
$$h(n) = 0, \quad n < N_2 > -\infty \qquad \text{.........} \qquad 1.1$$
$$\text{and } N_1 > N_2$$

2) Infinite Duration Impulse Response (IIR): This term means that the duration of the impulse response h(n) is infinite; i.e., there exists no finite values of either N_1 or N_2 such that (1.1) is satisfied.

3) Recursive realization: This term describes the way a filter (either IIR or FIR) is realized. It means that the current filter output y(n) is obtained explicitly in terms of past filter outputs y(n-1),.... as well as in terms of past and present filter inputs x(n), x(n-1),..... Thus the output of a recursive realization can be written as

$$y(n) = F(y(n-1), y(n-2), \ldots.., x(n), x(n-1), \ldots..)$$

4) Nonrecursive realization: This term means that the current filter output y(n) is obtained explicitly in terms of only past and present inputs; i.e., previous outputs are not used to generate the current output. The representation on a nonrecursive realization can be written as

$$y(n) = F(x(n), x(n-1), \ldots..)$$

It should be noted that, in general, recursive realizations of IIR filters and nonrecursive realizations of FIR filters are most efficient and are usually used.

1.3 SOME ADVANTAGES OF FIR FILTERS

Among the advantages of FIR filters, over IIR filters are,

1) FIR filters can easily be designed to approximate a prescribed magnitude frequency response to arbitrary accuracy with an exactly linear phase characteristic.

2) FIR filters can be realized efficiently both nonrecursively (using direct convolution[6], or high-speed convolution by using the fast Fourier transform[2]) and recursively (using a comb filter and a bank of resonators).

3) An FIR filter realized nonrecursively is always stable. FIR filters realized non-recursively contain only zeros in the finite z plane [illegible] and hence are always stable.

4) Quantization and roundoff problems inherent in recursive realization of IIR filters are generally negligible in nonrecursive realization of FIR filters.

5) The coefficient accuracy problems inherent in sharp cutoff IIR filters can often be made less severe for realizations of equally sharp FIR filters.

1.4 DESIGN OF FIR FILTERS USING WINDOWS

The most straightforward approach to FIR filter design is to obtain a finite length impulse response by truncating an infinite-duration impulse response sequence. If we suppose that $H_d(e^{jw})$ is an ideal desired frequency response, then

$$H_d(e^{jw}) = \sum_{n=-\infty}^{\infty} h_d(n)\ e^{-jwn} \quad \ldots\ldots \qquad 1.2$$

where $h_d(n)$ is the corresponding impulse response sequence, i.e.,

$$h_d(n) = ½ \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{jw})\ e^{jwn}\ dw \quad \ldots\ldots\ldots \qquad 1.3$$

In general, $H_d(e^{jw})$ for a frequency selective filter may be piecewise constant with discontinuities at the boundaries between bands. In such cases the sequence $h_d(n)$ is of infinite duration and it must be truncated to obtain a finite-duration impulse response Equations (1.2) & (1.3) can be thought of as a Fourier series representation of the periodic frequency response $H_d(e^{jw})$, with the sequence $h_d(n)$ playing the role of the 'Fourier coefficients". In order to control the convergence of the Fourier series a weighting function is used to modify the Fourier coefficients. This time-limited weighting function is called a window.

If $h_d(n)$ has infinite duration, one way to obtain a finite - duration causal impulse response is to simply truncate h(n), i.e. defining

$$h(n) = \begin{cases} hd(n), & 0 \leq n \leq N-1 \\ 0, & \text{Otherwise} \end{cases} \quad \ldots\ldots \qquad 1.4$$

In general, we can represent h(n) as the product of the desired impulse response and a finite-duration "Window" w(n); i.e.,

$$h(n) = h_d(n)\ w(n)$$

where in the example of Eq. (1.4),

$$w(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{Otherwise} \end{cases} \quad \ldots\ldots \quad (1.5)$$

Using the complex convolution theorem[10]

$$H(e^{jw}) = ½ \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\ W(e^{j(w-\theta)})\ d\theta \ \ldots\ldots \quad 1.6$$

That is, $H(e^{jw})$ is the periodic continuous convolution of the desired frequency response with the Fourier transform of the window. Thus the frequency response $H(e^{jw})$ will be a "smeared" version of the desired response $H_d(e^{jw})$. Figure 1.1(a) depicts typical functions $H_d(e^{j\theta})$ and $W(e^{jw-\theta})$ as required in Eq. (1.6).

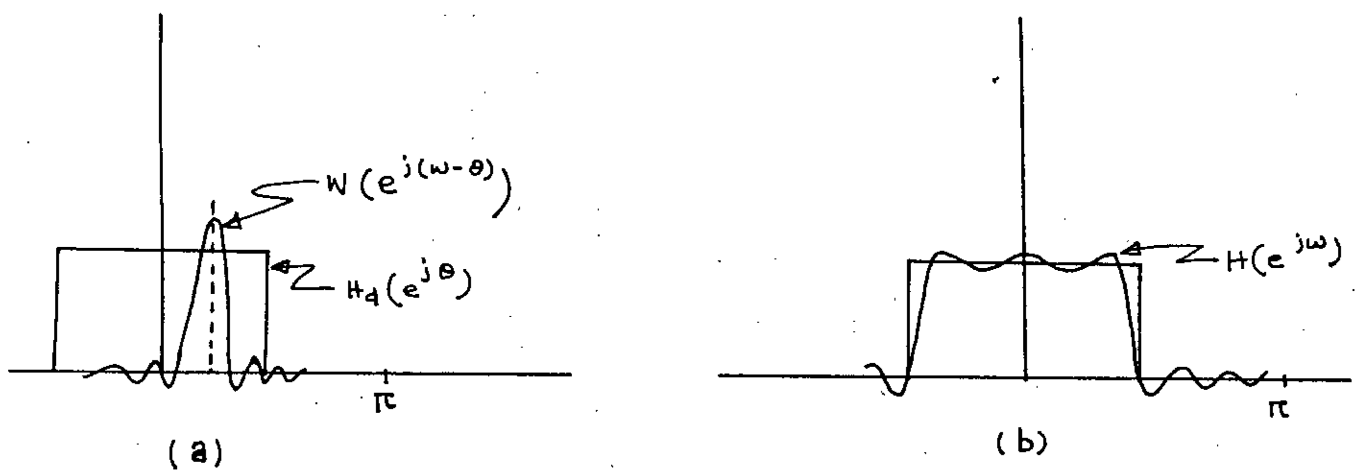


Fig. 1.1. (a) Convolution process implied by truncation of the desired impulse response;

(b) Typical approximation resulting from windowing the derived impulse response.

From Equation (1.6) it is clear that if $W(e^{jw})$ is narrow compared to variations in $H_d(e^{jw})$, then $H(e^{jw})$ will "look like" $H_d(e^{jw})$. Thus the choice of window is governed by the desire to have w(n) as short as possible in duration so as to minimize computation in the implementation of the filter and to faithfully reproduce the desired frequency responses.

1.5 COMPUTER-AIDED DESIGN OF FIR FILTERS

The window technique is straightforward to apply and is in a sense quite general. However, we often wish to design a filter that is the "best" that can be achieved for a given value of N. In the case of window designs, it follows from a fundamental result of the theory of Fourier series that the rectangular window provides the best mean - square approximation to a desired frequency response for a given value of N. That is

$$h(n) = \begin{cases} h_d(n), & 0 \leq n \leq N-1 \\ 0, & \text{Otherwise} \end{cases}$$

minimizes the expression -

$$E^2 = \tfrac{1}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_d(e^{jw}) - H(e^{jw})|^2 dw$$

However, it is found that this approximation criterion leads to adverse behavior at discontinuities of $H_d(e^{jw})$. A better criterion for many types of filters is minimization of the maximum absolute least square error. In our method of designing the FIR filters we used an optimization technique governed by a pattern search iterative procedure to minimize the least square error in one or more frequency bands. This optimization technique yielded better filters than the window method at the expense of greater complexity in the design procedure.

1.6 LITERATURE SURVEY

Since World War II, electronics engineers have speculated on the applicability of digital hardware techniques to many problem areas in which signal processing plays a role. Thus, for example, Laemmel (1948) in Bell Telephone Laboratory tried to employ digital elements to construct a filter. Needless to say, the conclusion then was not favourable. Cost, size, and reliability strongly favoured analog filtering and analog spectrum analysis techniques. In the 1950's, Stockham (1955) reports that Linville, an MIT professor at that time, discussed digital filtering at graduate seminars. By then, control theory, based partly on Hurawiez's (1945) work, had become established as a discipline; the concepts of sampling and its spectral effects were well understood and the mathematical tools of z-transformation theory, which had existed since Laplace's time, were propagating into the electronics engineering community. It was not until the mid 1960's that a more formal theory of digital signal processing began to emerge. By then the potential of integrated circuit technology was appreciated and it was not unreasonable to imagine complete signal processing systems that could best be synthesized with digital components.

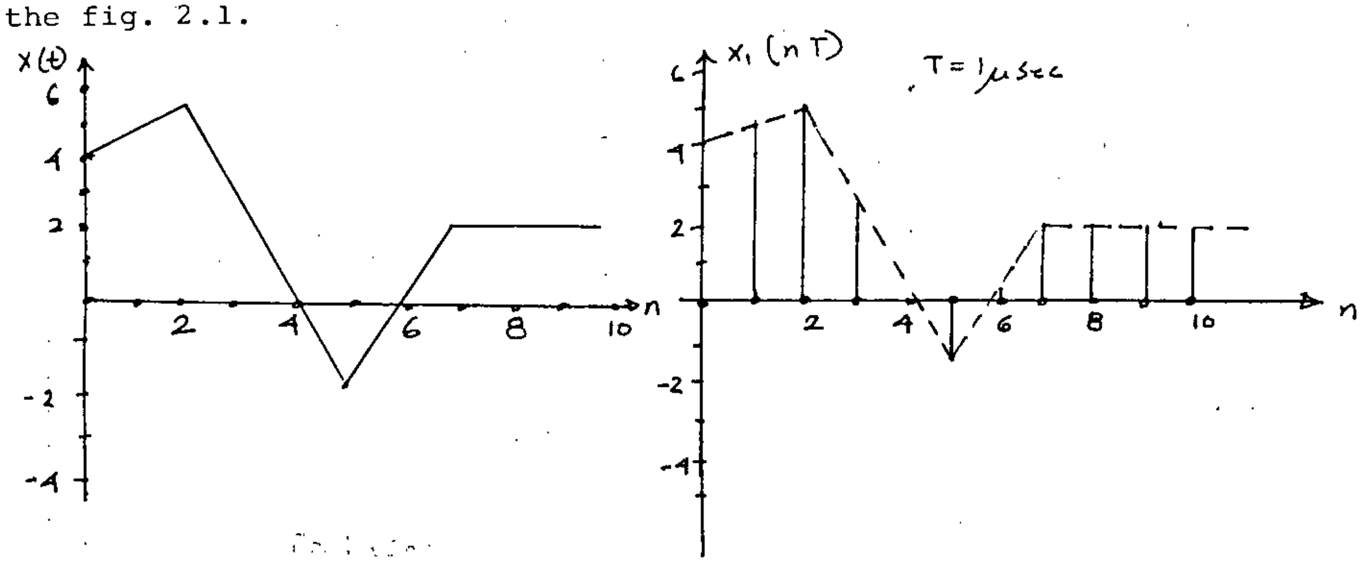
The first major contributions in the area of digital filter design were by kaiser (1966) at Bell Laboratories. Kaiser's work showed clearly how to design useful digital filters using the bilinear transform. Perhaps the most interesting aspect of the development of the field of digital signal processing is the changing relationship between the roles of FIR and IIR digital filters. Initially Kaisor (1970) analyzed FIR filters using window functions, which indicated that IIR filters were much more efficient than FIR filters. However, Stockham's work on the FFT (Fast Fourier Transform) method of performing convolution, or more specifically FIR digital filtering, indicated that implementation of high-order FIR filters could be made extremely computationally efficient. These results also inspired significant research for efficient designs for FIR filters.

CHAPTER 2

DIGITAL SYSTEMS

2.1 TYPES OF DISCRETE-TIME SIGNALS

A continuous-time signal can be represented by a function x(t) whose domain is a range of numbers (t_1, t_2), where $-\infty \leqslant t_1$ and $t_2 \leqslant \infty$ Similarly, a discrete-time signal can be represented by a function x(nT), where T is a constant and n is an integer in the range (n_1, n_2) such that $-\infty \leqslant n_1$ and $n_2 \leqslant \infty$. Alternatively, a discrete-time signal can be represented by x(n). As for continuous-time signals, two types of discrete-time signals can be identified, namely nonquantized and quantized signals. A nonquantized signal is one that can assume any value within a specified range, whereas a quantized signal is one that can assume only a finite number of discrete values. The various types of signals are illustrated in the fig. 2.1.



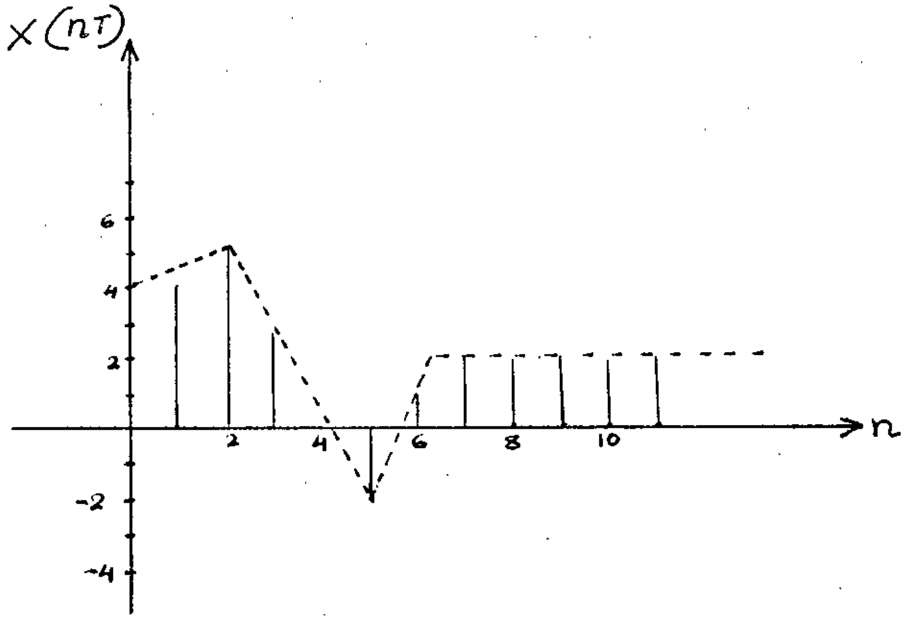


Fig. 2.1:
Legend X(t) = continuous-time signal, input to samplar

$X_1(nT)$ = discrete-time signal (nonquantized), output of the samplar.

X(nT) or X(n) = digital signal, output of the quantizer.

2.2a. REPRESENTATION OF ARBITRARY SYSTEMS

Using the basic digital impulse sequence it is easy to represent arbitrary sequence in terms of delayed and scaled impulses. For example, the sequence P(n) of fig 2-2 can be expressed as

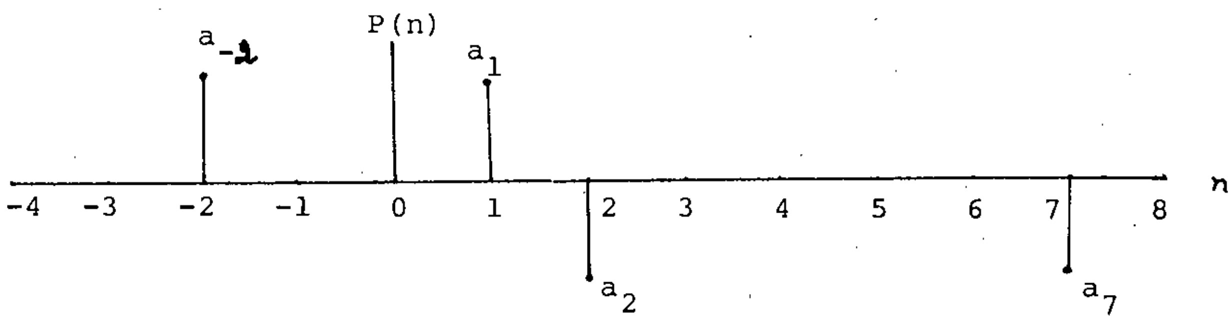


Fig. 2.2. Example of an arbitrary system

$$P(n) = a_{-2}\delta(n+2) + a_1\delta(n-1) + a_2\delta(n-2) + a_7\delta(n-7)$$

More generally, an arbitrary sequence can be expressed as

$$X(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \quad .... \qquad 2.1$$

2.2b LINEAR SHIFT-INVARIANT SYSTEMS

A system is defined mathematically as a unique transformation or operator that maps an input sequence x(n) into an output sequence y(n). This is denoted as

$$y(n) = T[x(n)] \quad \ldots\ldots\ldots \quad 2.2$$

and is often depicted as in fig. 2.3.

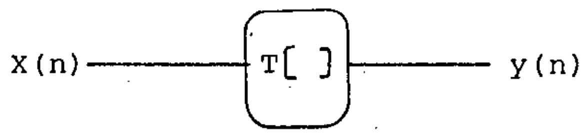


Fig. 2.3, Representation of the transformation that maps an input sequence x(n) into an output sequence y(n).

The class of Linear systems is defined by the principle of superposition. If $y_1(n)$ and $y_2(n)$ are the responses when $x_1(n)$ and $X_2(n)$, respectively, are the inputs, then a system is linear, if and only if

$$T(aX_1(n)+bX_2(n)) = aT(x_1(n))+bT(X_2(n)) = ay_1(n)+by_2(n) \ldots \quad 2.3$$

where, a and b are arbitrary constants.

Linear system can be completely characterized by its impulse response or unit sample response. Specifically, let $h_k(n)$ be the response of the system to $\delta(n-k)$, a unit sample occuring at n=k. Then from eqn. 2.1.

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} X(K)\,\delta(n-k)\right]$$

$$\text{Or } y(n) = \sum_{k=-\infty}^{\infty} X(k)\ T\left[\delta(n-k)\right]$$

$$\text{Or } y(n) = \sum_{k=-\infty}^{\infty} x(k)\ h_k(n)\ \ldots\ldots \qquad 2.4$$

Thus according to Eq. 2.4, the system response can be expressed in terms of the response of the system to $\delta(n-k)$.

The class of shift-invariance systems (Time invariant system) is characterized by the property that if y(n) is the response to x(n), then y(n-k) is the response to x(n-k), when k is a positive or negative integer.

This property of shift invariance implies that if h(n) is the response to $\delta(n)$, then the response to $\delta(n-k)$ is simply h(n-k). Thus, Eq. 2.4, becomes,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)\ \ldots\ldots \qquad 2.5$$

Equation(2.5) is commonly called the convolution sum. It can be represented as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(K)\ x(n-k)\ \ldots\ldots \qquad 2.6$$

2.3 THE GENERAL DIFFERENCE EQUATION DESCRIBING THE DIGITAL FILTER

Let the sequence of numbers x(n) be such a set of equally spaced measurements of some quantity x(f), where n is an integer and t is a continuous variable. If the sequence y(n) is computed by the formula,

$$\sum_{k=o}^{N} b_k y(n-k) = \sum_{r=o}^{M} a_r x(n-r) \ldots\ldots \qquad 2.7$$

We can write

$$y(n) = \sum_{k=o}^{M} C_k \, x(n-k) + \sum_{k=1}^{N} d_k \, y(n-k) \ldots. \qquad 2.8$$

Then this formula defines a digital filter. The coefficients a_r, b_k, c_k & d_k are constants. Thus a digital filter is merely a linear combination of equally spaced samples x(n-k) of some function X(t), together with the computed values of the output y(n-k). For each successive n, the formula shifts one data point along the string of data points, x(n-k).

In the case where all the coefficients d_k of the y(n-k) are zero, the filter is called nonrecursive; otherwise it is a recursive filter. According to the duration of the impulse response the digital filters are classified into another two classes. If the impulse response is of finite duration, it is reffered as a finite impulse response (FIR) Filter, and if the impulse response is of infinite duration, it is reffered as an infinite impulse response (IIR) filter. The FIR filters are the nonrecursive type of digital fillers i.e. the constants d_k of Eq. 2.8 are zero. So, the equation describing FIR filter is,

$$y(n) = \sum_{k=o}^{M} C_k \, x(n-k) \ldots\ldots. \qquad 2.9$$

If fact, comparison with Eq. (2.6) shows that the above difference equation is identical to the convolution sum, and hence if follows directly that.

$$h(n) = \begin{cases} C_n \, , & n = 0,1, \ldots\ldots, M \\ 0, & \text{Otherwise.} \end{cases}$$

2.4 DERIVATION OF THE GENERAL TRANSFER FUNCTION OF DIGITAL FILTERS

Let X(n) be the input to a linear time-invariant digital system, with an impulse response h(n), Let X(z) and H(z) be the z-transforms of x(n) and h(n), respectively. The output y(n) is given by

$$y(n) = x(n) * h(n)$$

Applying the transform property (Appendix A)

$$Y(z) = H(z)\ X(z)$$

where H(z) is the transfer function of the system. For deriving the General transfer function of the degital filter. Let us start with the general equation (Eq. 2.7),

$$\sum_{k=o}^{N} b_k\, y(n-k) = \sum_{r=0}^{M} a_r\, x(n-r)$$

Applying the z - transform to each side we have

$$z\left[\sum_{k=o}^{N} b_k\, y(n-k)\right] = z\left[\sum_{r=o}^{M} a_r\, x(n-r)\right]$$

which from property of Appendix B in the Appendix A we can rewrite as,

$$\sum_{k=o}^{N} b_k \, z\left[y(n-k)\right] = \sum_{r=o}^{M} a_r \, z\left[x(n-r)\right]$$

with the help of property 2 of Appendix B, that,

$$z\left[y(n-k)\right] = z^{-k} \, Y(z)$$

$$\text{and } z\left[x(n-r)\right] = z^{-r} X(z)$$

Thus,

$$\sum_{k=o}^{N} b_k \, z^{-k} \, Y(z) = \sum_{r=o}^{M} a_r \, z^{-r} \, X(z)$$

$$\text{Or, } H(z) = \sum_{r=o}^{M} a_r \, z^{-r} \Big/ \sum_{k=o}^{N} b_k \, z^{-k} \quad \ldots\ldots \qquad 2.10$$

For FIR filter, we have, N=0,

$$\text{Or } H(z) = \sum_{r=o}^{M} a_r \, z^{-r} \quad \ldots\ldots \qquad 2.11$$

$$= a_o + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_m z^{-m} \ldots \qquad 2.12$$

The z transfer of a sequence is always a rational function of either z or z^{-1} (App-B). Thus, if we know the poles and zeros of the z-transform H(z) of a sequence of inpulse response h(n), we can write H(z) in the following form.

$$H(z) = \frac{\alpha \prod_{i=1}^{M} (1 - z_i z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} \quad \ldots\ldots \qquad 2.13$$

where, z_i are the zeros and P_k are the poles of the transfer function H(z) in z domain.

Equation 2.13 is another general form of transfer function of digital filters

2.5 CONVERSION FROM Z-DOMAIN TO FREQUENCY DOMAIN.

The Fourier transform $X_F(e^{jw})$ of a sequence x(n) is defined to be,

$$X_F(e^{jw}) \triangleq \sum_{n=-\infty}^{\infty} x(n)\ e^{-jnw} \quad \ldots\ldots \qquad 2.14$$

Comparing this equation with defining equation of z - transformation (Appendix A), we can conclude that the Fourier transform of a sequence is the z-transform of the sequence evaluated along the unit circle in the z-plane (Appendix A), as shown in Fig 2.4.

That is

$$X_F(e^{jw}) = X(z)\Big|_{z=e^{jw}} \triangleq X(e^{jw})$$

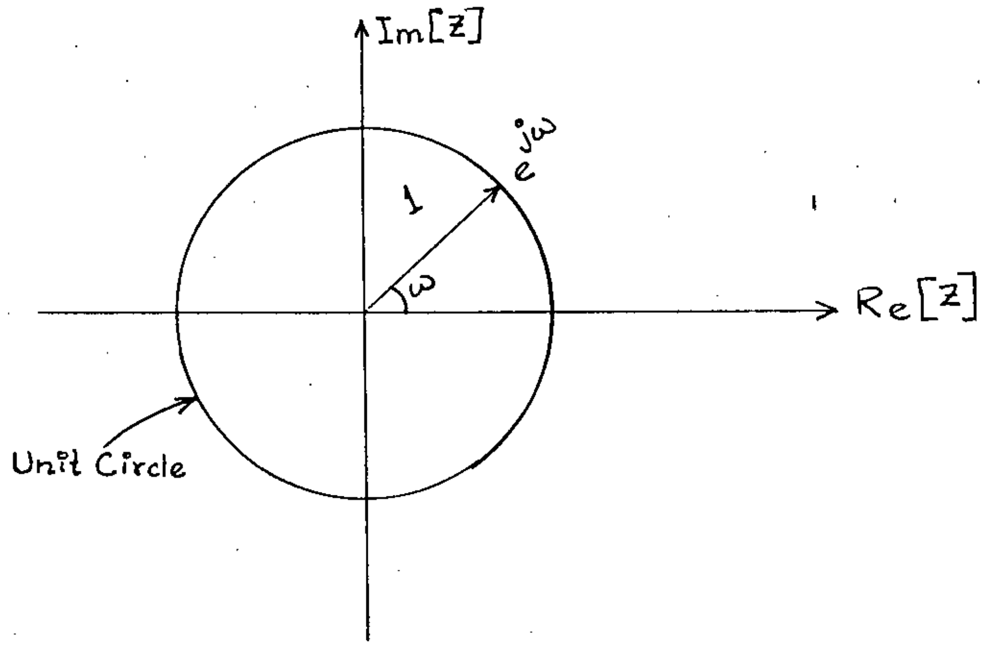


Fig. 2.4. Fourier transforms are z-transforms evaluated along the unit circle.

For any integer K, we have,

$$e^{jw} = e^{j(w+2\pi k)}$$

Hence, $X(e^{jw}) = X\left[e^{j(w+2\pi k)}\right]$

This means that the Fourier transform of a sequence is a periodic function of w.

Given the input sequence x(n), the output sequence y(n) can be obtained by a convolution sum, as indicated by (2.6).

Let that x(n) is given by

$$X(n) = e^{jwn}, \quad -\infty < n < \infty$$

Then the output y(n) is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\ x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\ e^{jw(n-k)}$$

$$= e^{jwn} \sum_{k=-\infty}^{\infty} h(k)\ e^{-jwk}$$

$$= H(e^{jw})\ e^{jwn}$$

$$= H(e^{jw})\ X(n) \quad \ldots\ldots\ldots \qquad 2.15$$

where w is the input exponential frequency. [Let $x(t) = e^{jwt}$ be the exponential sinusoidal continuous - time function. Then the corresponding sampled sequence x(n) is given by $x(n) \triangleq x(nT) = e^{jwnT} = e^{jwn}$, where $w \triangleq wT$ is the digital frequency of the sampled sequence x(n), and T is the sampling period] From (2.15), $H(e^{jw})$ is the multiplier to that converts the exponential input sequence x(n) to the output sequence y(n). $H(e^{jw})$ is called the frequency response of the system. In otherwords, the frequency response $H(e^{jw})$ of a system S is the transfer function H(z) of the system S evaluated along the unit circle - by letting $z = e^{jw}$ with $0 \leqslant w < 2\pi$ - in the z-plane.

2.6 LINEAR PHASE CHARACTERISTICS OF FIR FILTERS:

The transfer function of an FIR digital filter is in the form of

$$H(z) = \sum_{n=o}^{N-1} h(n) \; z^{-n} \; \ldots\ldots, \qquad 2.16$$

where the impulse response is of length N or has a duration of N samples. If the impulse response of an FIR digital filter satisfies.

$$h(n) = h(N-1-n) \; \ldots\ldots \qquad 2.17$$

for n = 0, 1, ... , (N/2) - 1 if N is even, and

for n = 0,1, ... , (N-1)/2 if N is Odd, then it can be shown that the digital filter will have a linear phase characteristic. Indeed, when N is odd, (2-16) and (2-17) give,

$$H(e^{jw}) = \sum_{n=o}^{N-1} h(n) \; e^{-jnw}$$

$$= \sum_{n=o}^{(N-3)/2} \left[h(n)\ e^{-jnw} + h(N-1-n) e^{-j(N-1-n)w} \right]$$

$$+ h\left(\frac{N-1}{2}\right) e^{-j[(N-1)/2]w}$$

$$= \sum_{n=o}^{(N-3)/2} h(n) \left[e^{-jnw} + e^{-j(N-1-n)w} \right]$$

$$+ h\left(\frac{N-1}{2}\right) e^{-j[(N-1)/2]w}$$

$$= e^{-j[(N-1)/2]w} \left\{ h\left(\frac{N-1}{2}\right) \right.$$

$$\left. + \sum_{n=o}^{(N-3)/2} h(n) \left[e^{-j\{n-[(N-1)/2]\}w} + e^{j\{n-[(N-1)/2]\}w} \right] \right\}.$$

$$= e^{-j[(N-1)/2]w} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=o}^{(N-3)/2} 2\ h(n)\ \mathrm{Cos}\left[\left(n-\frac{N-1}{2}\right)w\right] \right\}$$

In the same manner, when N is even, the frequency response is given by

$$H(e^{jw}) = e^{-j[(N-1)/2]w} \left\{ \sum_{n=o}^{(N/2)-1} 2h(n)\ \mathrm{Cos}\left[\left(n-\frac{N-1}{2}\right)w\right] \right\}$$

In both cases, the phase $\phi(w)$ of the FIR digital filter is given by

$$\phi(w) = -\underline{/H(e^{jw})} = \frac{N-1}{2}\omega \qquad \ldots\ldots \qquad 2.18$$

which is linear for $-\pi < w \leq \pi$. The group delay function

$$\tau(w) \triangleq \frac{d\,\phi(w)}{dw} = \frac{N-1}{2} \qquad \ldots\ldots \qquad 2.19$$

linear phase filter for N=10. (even) is shown in fig (2-6). Here $\gamma(w)=4.5$, which reflects that the center of symmetry of the impulse response lies midway between two samples. The definition of a Linear phase filter [Eq. 2.18] requires the filter to have both constant group delay and constant phase delay.

2.7 STABILITY CONSIDERATIONS

A linear and time-invariant digital system S is stable if its impulse response h(n) satisfies the condition

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \ldots\ldots\ldots\ldots \qquad 2.20$$

We will try to implement the above stability criterion in the case of digital filters in the z - domain. Since, in the case of FIR filter we know that the duration of the impulse response is finite, so we can say that FIR filters will always satisfy the above mentioned stability criterion. But to make it more specific and clear we will first study the stability condition of IIR filter.

Considering the general transfer function of the following type

$$H(z) = \frac{\alpha \prod_{i=1}^{M} (1-z_i z^{-1})}{\prod_{k=1}^{N} (1-P_k z^{-1})} \quad \ldots\ldots \qquad 2.21$$

Any filter whose transfer functions are given by (2.21) with $N \geqslant 1$ is called an IIR digital filter, because there does not exist a finite integer L such that, h(n) = 0, for $n > L$, where h(n) is the impulse response of the filter.

Considering, $M \leq N$, a partial fraction expansion of (2.21) gives.

$$H(z) = \xi_0 + \frac{\xi_1}{1-P_1 z^{-1}} + \frac{\xi_2}{1-P_2 z^{-1}} + \ldots\ldots + \frac{\xi_N}{1-P_N z^{-1}}$$

where, $\xi_i = (1-P_i z^{-1}) H(z) \Big|_{z=P_i}$ for i=1,2, ... N.

and P_i are the poles of (2.21).

Hence, the corresponding impulse response of (2.21) with the help of inverse. Z-transform is

$$h(n) = \left[\xi_1 P_1^n + \xi_2 P_2^n + \ldots\ldots + \xi_N P_N^n\right] u(n)$$

$$+ \xi_0 \delta(n) \quad \ldots\ldots \tag{2.22}$$

Clearly, the necessary and sufficient conditions for the impulse response of (2.21) to satisfy the stability criteria of Eq. (2.20) is that, $|P_i| < 1$ for i = 1,2, ..., N. That is all the pole locations of the digital filter are within the unit circle in the z-plane.

We know that the Transfer function of FIR digital filter does not carry any poles. Hence, this type of filter is always stable.

CHAPTER - 3

REALIZATION OF FIR FILTER

3.1 Introduction

As stated before, a digital filter can be implemented either as software on a general or special - purpose computer or as hardware. Either way, the basic concepts of digital filter implementation involve the following two steps:

1. to convert the input-output relationship of a digital filter into an algorithm.
2. to implement or to realize the algorithm in terms of a set of basic operations or digital hardware.

In this chapter a direct form approach of realization of FIR filters is suggested.

3.2 Basic Building Block Considerations

We know the FIR filter transfer function can be written in the following form from Eq. (2-).

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{n=o}^{N-1} h(n)\ z^{-n} \ldots$$

$$\text{Or, } H(z) = a_o z^o + a_1 z^{-1} + a_2 z^{-2} \ldots\ldots + a_M\ z^{-M} \ldots\ldots \qquad 3.1$$

where X(z) and Y(z) are, respectively, the z-transforms of the input and the output sequences. To realize this transfer function, We convert it into a difference equations as

$$y(n) = a_o\ x(n) + a_1\ x(n-1) + a_2\ x(n-2) \ldots\ldots$$

$$+ a_m\ x(n-M) \ldots\ldots \qquad 3.2$$

To realize the transfer function of (3.1) is equivalent to implementing the algorithm of (3.2), which requires the following:

a) delay units or shift registers to store past output values;

b) multipliers or multiplication operations to provide the necessary scaling or weighting factors to sampled values; and

c) summers or addition operations to add up the various quantities indicated on the right-hand side of (3.2) to give the present output values.

Fig. 3.1 illustrates the symbols and operations of these basic building blocks.

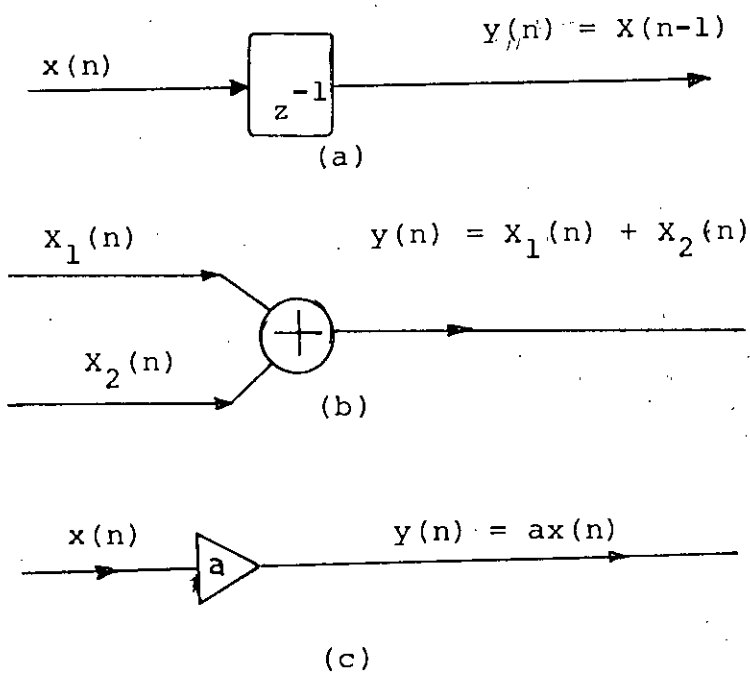


Fig. 3.1. Basic building blocks for digital filters: a) delays, b) summers and c) multipliers.

A simplified symbol representation of these building blocks is illustrated in the figure (3.2).

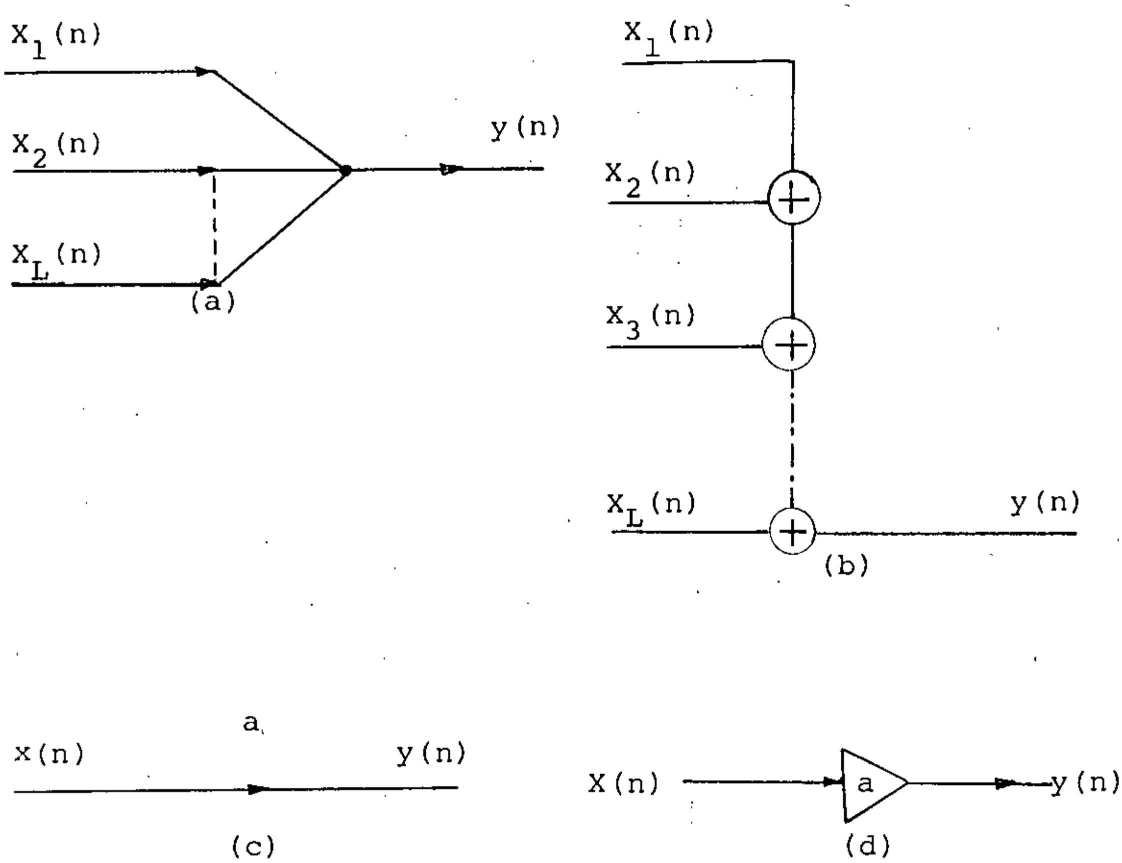


Fig. (3.2) Simplified schematics for (a) and (b) summers, and (c) and (d) multipliers.

3.3 Direct Realization

There are many techniques in the direct approach to realize digital transfer functions. Among the well-known techniques are the direct forms, ladder and lattice structures, multiplier - extraction techniques, and the modular forms of wave digital filters. The direct form is the realization technique that implement the difference equation of the filter. The multiplying constants are the coefficients of the transfer function. For low-order transfer functions, the direct form is very competitive in performance and cost.

A realization of the transfer function of (3.1) can be obtained by implementing a computation procedure for (3.2). Figure 3.3 shows a digital network implementation of (3.2). This configuration is called the direct form realization of the transfer function of (3.1).

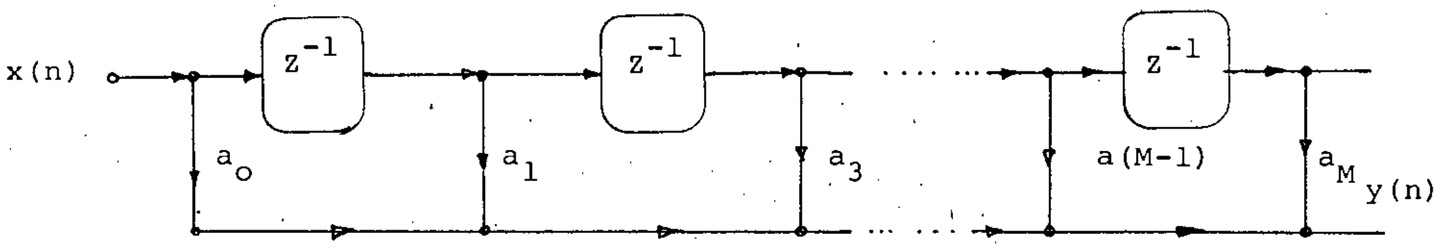


Fig. 3.3. Realization of FIR filters by Direct approach.

To analyse the hardware implementation we consider a filter with impulse response duration of 7 i.e. N = 7. The direct form realization is illustrated in the figure 3.4.

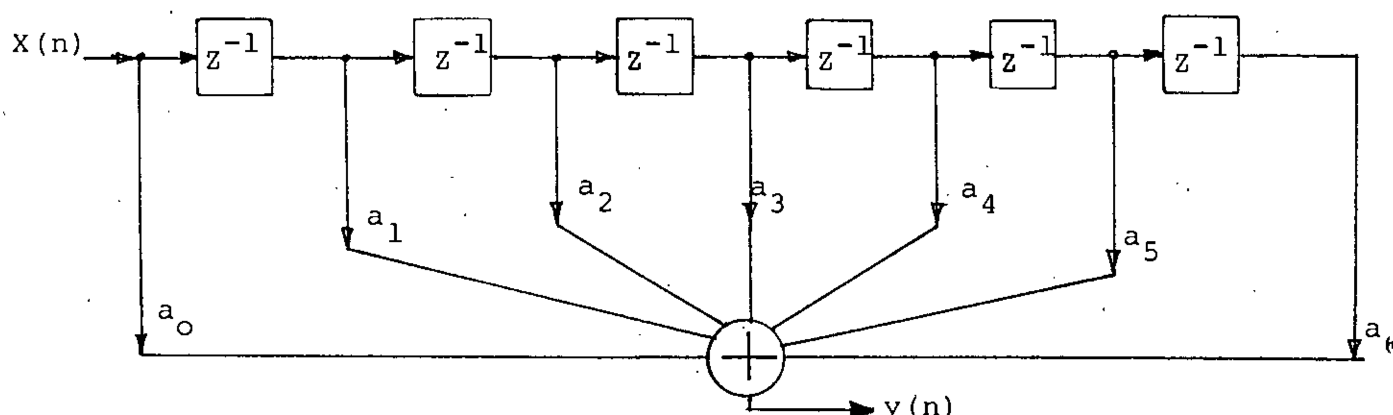


Fig. 3.4. Direct form FIR filter with N=7.

Figure 3.5 shows a simple structure for realizing the filter of Fig. 3.4 using a single computational element (consisting of a multiplier and an adder), a shift resister to hold the filter states, and a ROM for the coefficients. The important element of this figure is the manner in which the shift register is controlled. By means of a multiplier and an accumulator, a single output sample can be computed by successive addition as the shift register circulates. During the first computation (of a_6 x(n-6)), the new input x(n) inters the shift register while x(n-6) is shifted off the end. Afterward, each iteration includes a circulation of one datum around the shift register, as indicated in the chart accompanying Fig. 3.5. When y(n) is obtained, it is sent on while the accumulator is cleared; then the next major cycle begins.

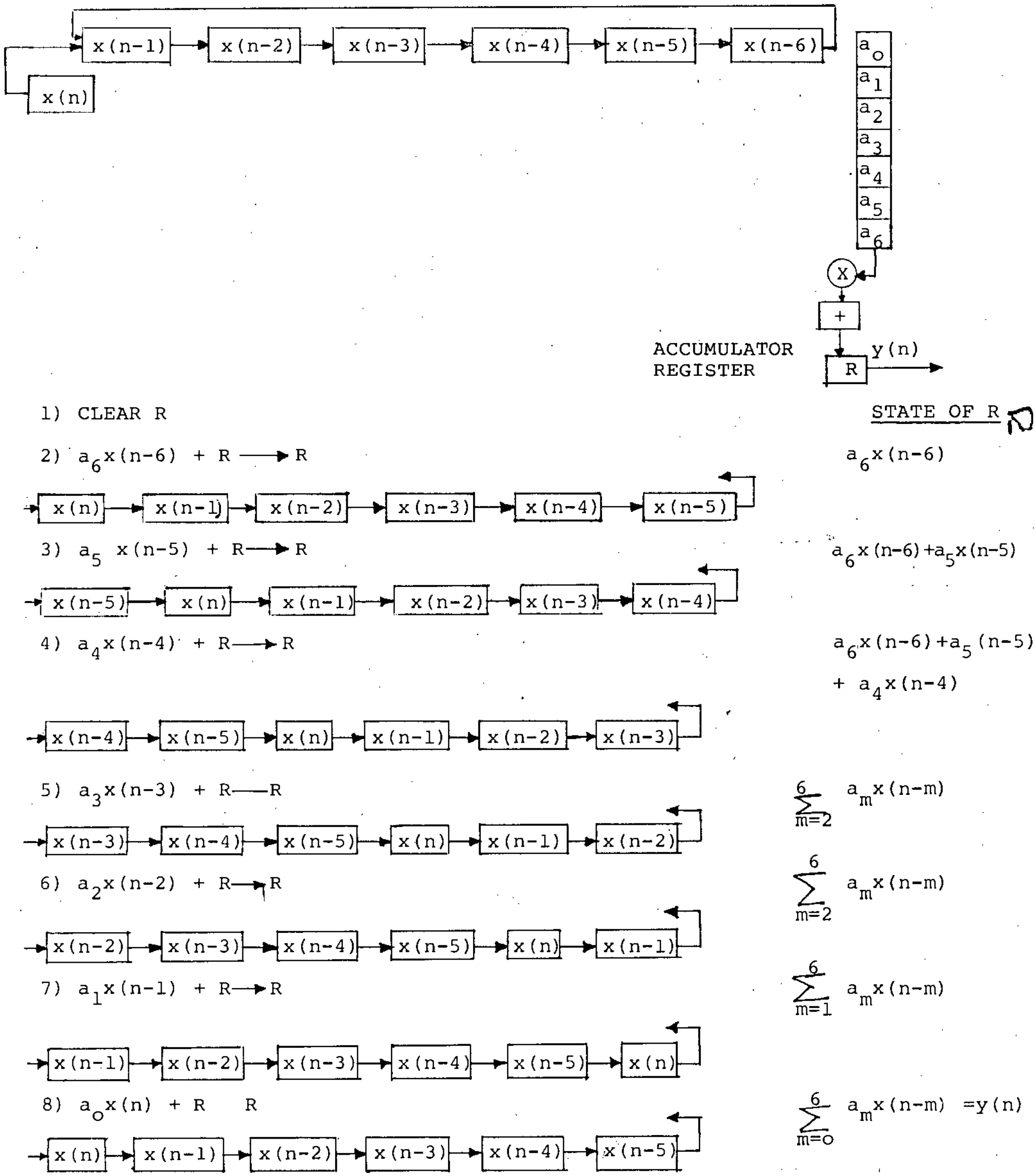


Fig. 3.5. Structure, program, and states for implementation of direct form FIR filter.

CHAPTER 4

DESIGN BY OPTIMIZATION TECHNIQUE

4.1 Introduction

Many work have been done on the design techniques of FIR digital filters, the objective of most of the work is to develop a method for building up a filter which approximates the desired characteristics very closely. For example the simplest method of windowing design is a common one. There are also other methods such as discrete fourier transform and equiripple approximation. Applying all these design techniques the filter which is developed requires some compromise with its desired characteristic. In most of the cases the compromise is made with the transition region, which results in a loss of information in the transition frequency band.

In our design with the optimization technique, much care is taken to study the transition region and selecting some optimal transition points to yield relatively better filter with minimum deviations in the pass and stop bands. The purpose of optimization is to find the best possible solution among the many potential solutions for a given problem in terms of some effectiveness or performance criterion. In this computer aided design we used Hooke and Jeeves pattern search (direct search) iterative optimization algorithm[4]. Using this search technique the resulted filter is a better one with specified tolerance of ripple in the pass band and stop band with tolerable transition region, but with the expense of greater computer time.

4.2 HOOKE AND JEEVES PATTERN SEARCH OPTIMIZATION TECHNIQUE

Conceptually, the simplest type of search method is to change one variable at a time while keeping all the others constant until the minimum is reached. The algorithm consists of two major phases, an "exploratory search" around the base point and a "pattern search" in a direction selected for minimization. Figure 4.1 is a simplified information flow diagram of the direct search algorithm. This algorithm operates in the following manner. Initial values for all the elements of x must be provided, as well as an initial incremental change Δx. To initiate an exploratory search, f(x) is evaluated at a base point (the base point is the vector of initial guesses of the independent variables for the first cycle). Then each variable is changed in rotation, one at a time, by incremental amounts, until all the parameters have been so changed. To be specific, $x_1^{(o)}$ is changed by an amount $+\Delta x_1^{(o)}$, so that $x_1^{(1)} = x_1^{(o)} + \Delta x_1^{(o)}$. If f(x) is reduced, $x_1^{(o)} + \Delta x_1^{(o)}$ is adopted as the new element in x. If the increment fails to improve the objective function, $x_1^{(o)}$ is changed by $-\Delta x_1^{(o)}$, and the value of f(x) again checked as before. If the value of f(x) is not improved by either $x_1^{(o)} \pm \Delta x_1^{(o)}$, $x_1^{(o)}$ is left unchanged. Then $x_2^{(o)}$ is changed by an amount $\Delta x_2^{(o)}$, and so on, until all the independent variables have been changed to complete one exploratory search. For each step or move in the independent variable, the value of the objective function is compared with the value at the previous point. If the objective function is improved for the given step, then the new value of the objective function replaces the old one in the testing. However, if a perturbation is a failure, then the old value of f(x) is retained.

After making one (or more) exploratory searches in this fashion, a "pattern search" is made in the following way,

$$x_i^{(k+1)} = 2x_i^{(k)} - x_i^{(b)},$$

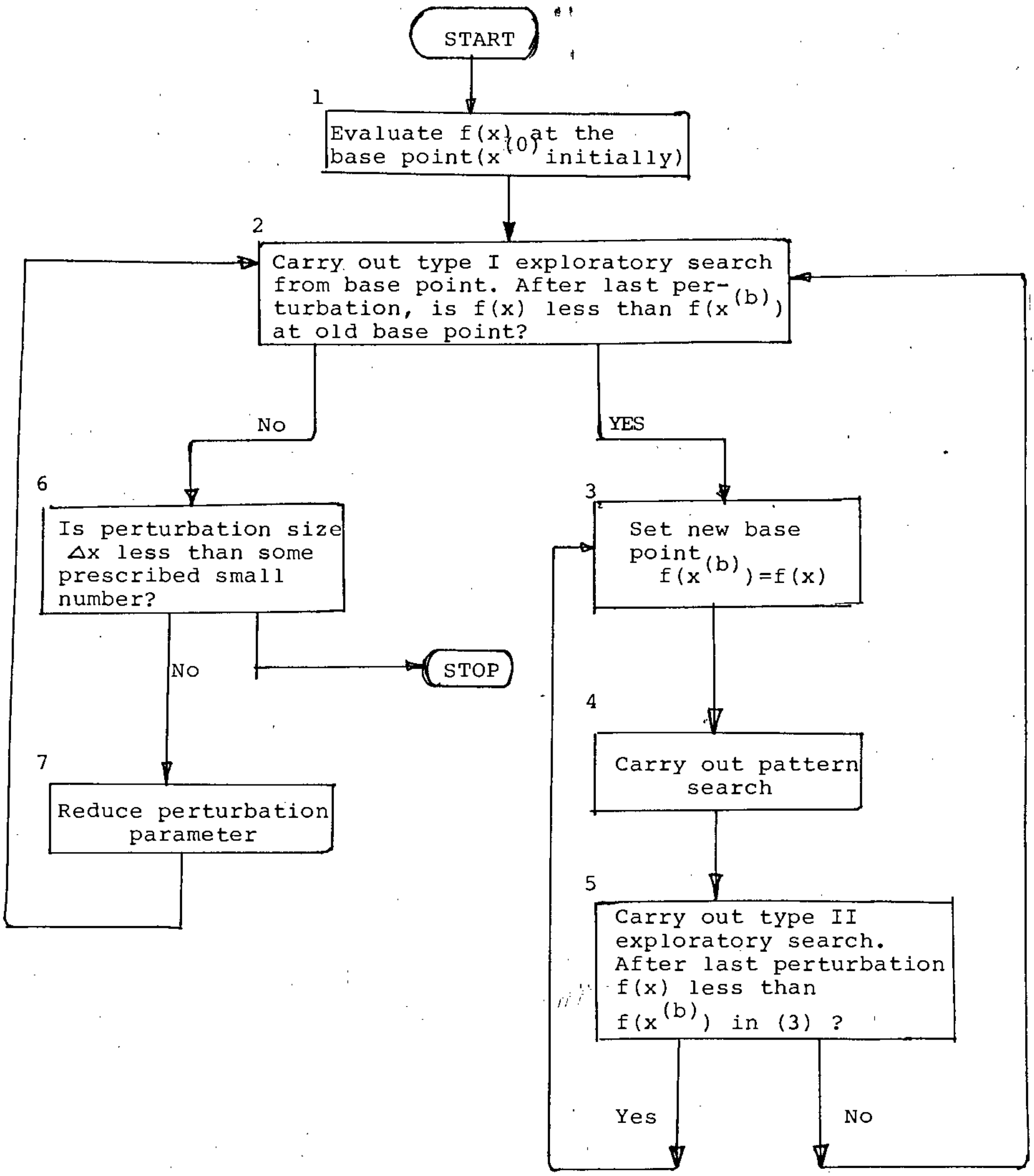
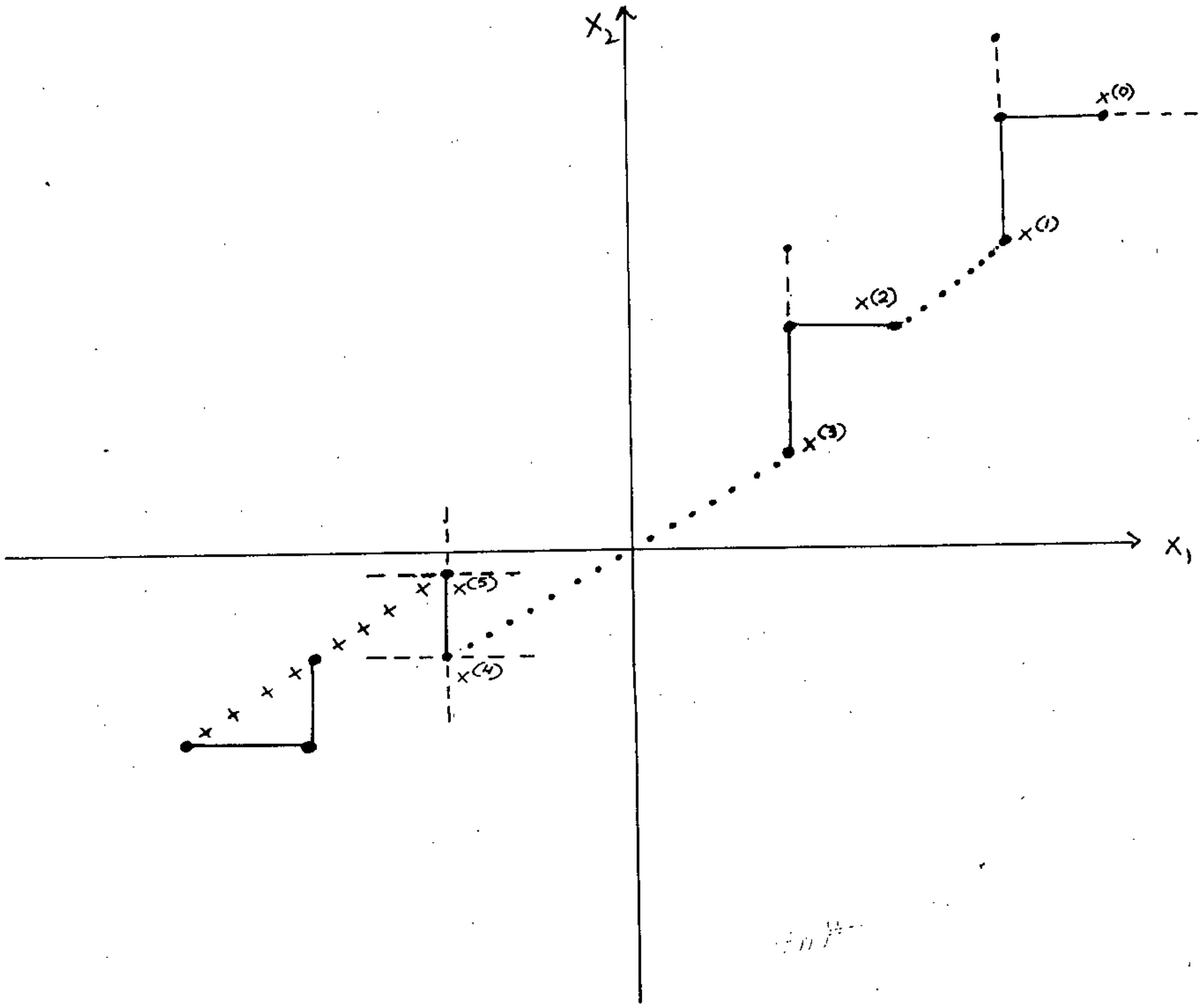


Fig 4.1. Information flow diagram for pattern search minimization.

where, $x_i^{(k+1)}$ is the new pattern search vector, $x_i^{(b)}$ is the old base vector and $x_i^{(k)}$ is the new base vector after 1st successful type I exploratory search is obtained. After the successful pattern search, an exploratory search is conducted which is known as type II exploratory search, and the success or failure of a pattern move is not established until after the type II exploratory search has been completed. If f(x) is not decreased after the type II exploratory search, the pattern search is said to fail, and a new type I exploratory search is made in order to define a new successful direction. If the type I exploratory search fails to give a new successful direction, Δx is reduced gradually, until either a new successful direction can be defined or each Δx_i becomes smaller than some preset tolerance. Failure to decrease f(x) for a very small Δx indicates that a optimum point has been reached. Figure 4.2, describes the whole procedure of searching operation.



● Base vectors $x^{(b)}$

Exploratory search steps

______ success

------- failure.

Pattern search steps

..... success

+++++ Failure.

Fig. 4.2. A simplified diagram explaining the steps of HOOKE & JEEVES optimization technique.

4.3 DESIGN OF THE FIR FILTER

4.3.1 SPECIFICATION OF THE DESIRED FILTER:-

To design a filter, the first step is to consider some desired specifications. In most of the cases much care is taken, so that the desired filter characteristics is matched closely with the ideal filter characteristics. Some parameters of the ideal filter characteristics are illustrated in the fig. 4.3.

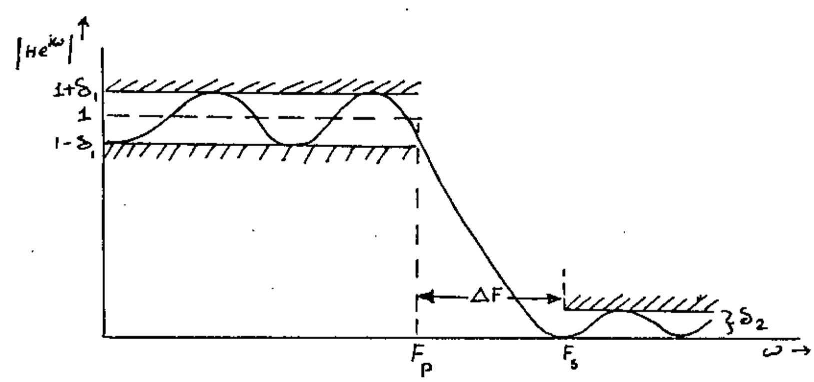


Fig. 4.3. Idealized frequency response(-------- curve)
Designed filter output frequency response(——— curve)

In the above figure, δ_1 and δ_2 are the permissible passband and stopband ripple respectively and F_p and F_s are the pass-band and stop-band cutoff frequencies. The other useful parameter indicated in the figure is the transition width ΔF which is defined as

$$\Delta F = F_S - F_P \qquad 4.1$$

Since from the beginning we are interested to design the filters with tolerable transition region, a compromise is made with the ripples, δ_1 and δ_2 in our design procedure. The ripples are the result of the well known Gribbs phenomenon. The reason for them is the presence of the sharp discontinuity in the transition region.

The design of an FIR filter may be accomplished by finding either its impulse response co-efficient or N samples of its frequency response. In our work we have concerned with the design techniques of low pass FIR filters only.

The design technique followed can be summerised as (1) to specify the desired characteristics of the filter to be designed and (2) to reach the filter equation for a given order which minimises a specified error function. To achieve the desired filter equation we have followed the optimisation technique (Hooke and Jeeves pattern search methods).

Fig (4.4) shows the ideal frequency response of a low pass filter in a frequency sampled form.

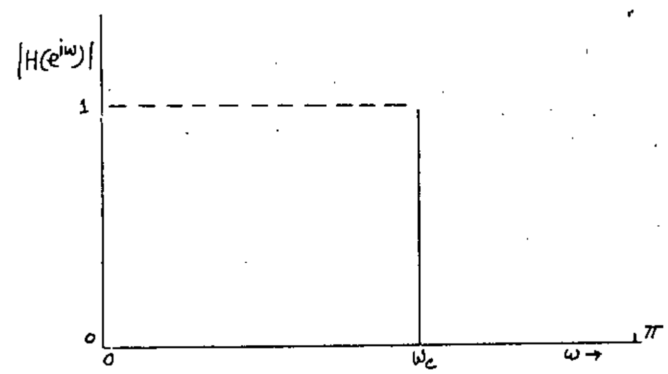


Fig (4.4) Fixed samples of ideal lowpass filter frequency response with no transition sample.

Mathematically the frequency response can be expressed as,

$$H_d(e^{jw}) = 1 \text{ for } o \leq w \leq w_c$$

$$= 0, \text{ otherwise}.$$

The equation of the FIR filter can be expressed as

$$H(z) = \sum_{n=o}^{N-1} h(n)z^{-n}$$

The solution of the design technique is to find out the values of h(n) for which the error function E(w) is minimised. In our present problem we have defined the error function as

$$E(w) = \sum \left\{ |H_d(e^{jw})| - |H(e^{jw})| \right\}^2 \quad \ldots\ldots \quad \ldots\ldots \quad 4.2$$

Another error function which is often used may be named as the maximum error in passband or stop band. To ensure the linear phase characteristics of the filter, a constrain has been imposed on the optimization such that, h(n) = h(N-1-n).

For a specification similar to the above mentioned one, a designed filter of the order N = 14 gives the characteristics as shown in figure(4.5a). The figure shows that the minimum stop band attenuation is about 20 dB, which is unsatisfactory for many purpose. This is a well known method to improve the minimum stop band attenuation by increasing the transition width. Figure (4.6) shows a specification where two samples H_1 and H_2 have been introduced in the transition region. The attenuation in the stop band depends upon the

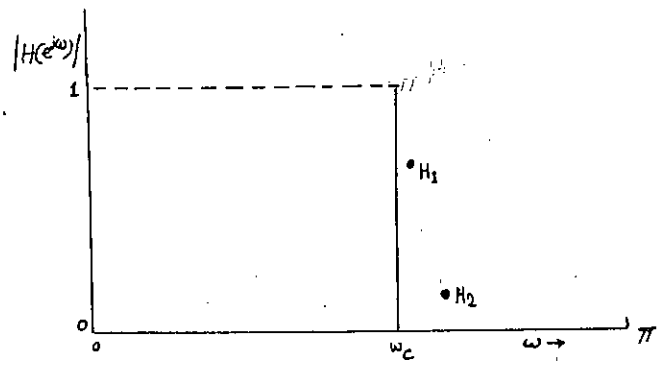


Fig. 4.6. Frequency sampling design using two transition samples.

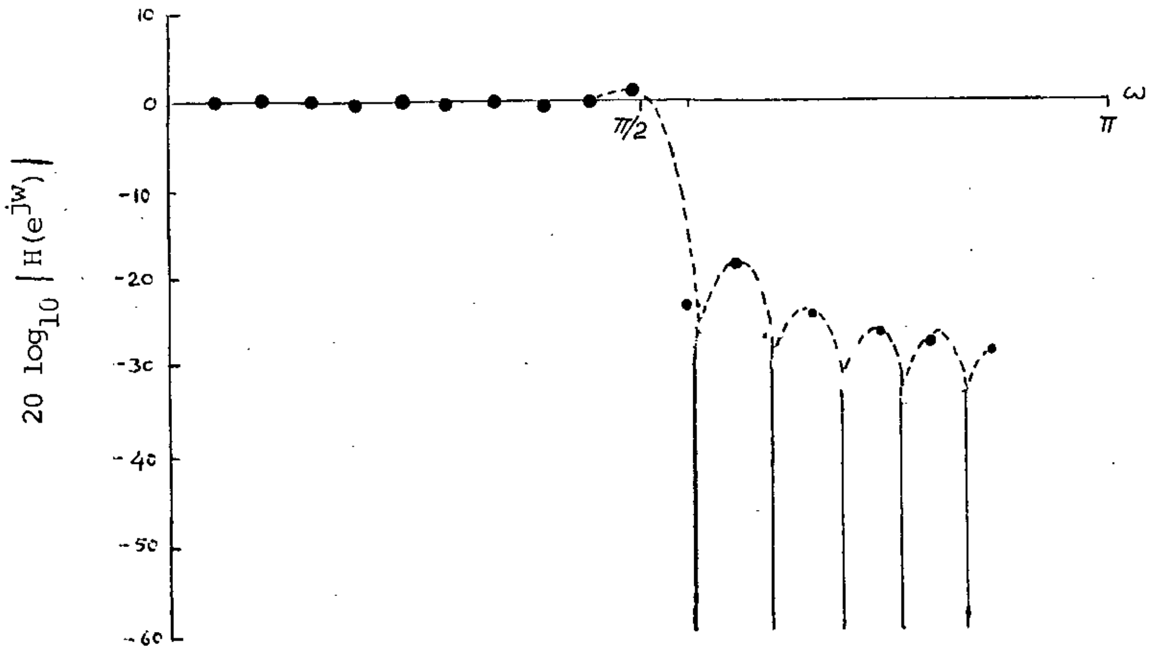


Fig. (4.5a). Frequency response of the filter without transition region samples (N=14).

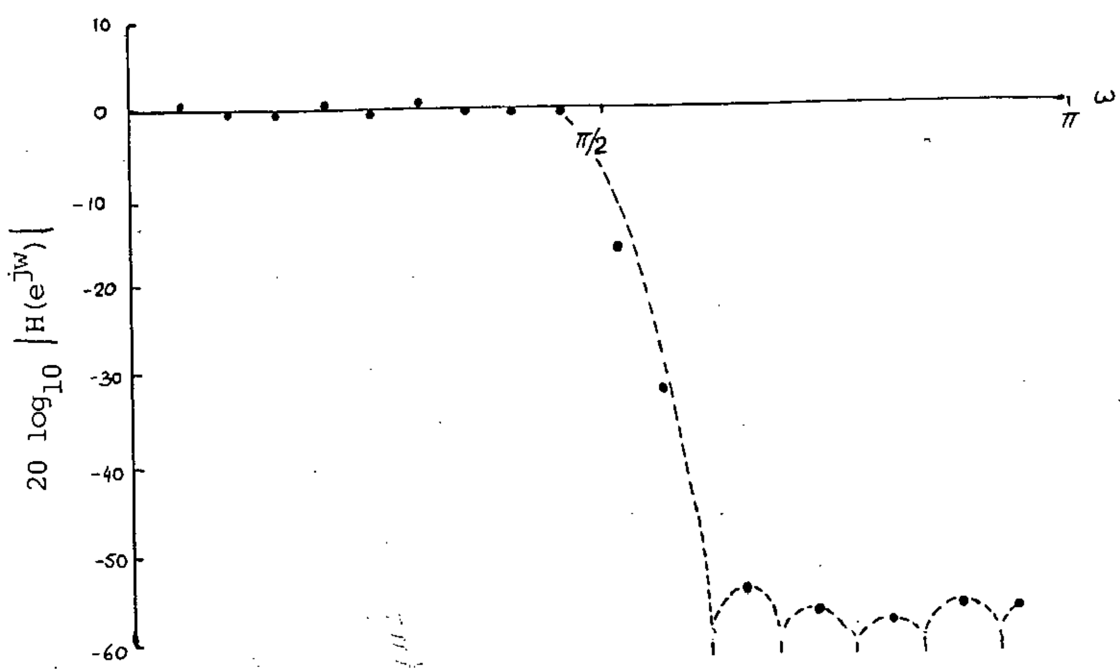


Fig. (4.5b). Frequency response of the filter with two transition samples (N = 14).

values of H_1 and H_2 of course it is also dependent upon the number of such parameters. Optimization technique can be used to vary these parameters so as to give the best approximation to the desired filter i.e. to improve maximally the minimum attenuation in the stop band. In our design process we have devided the transition region into three equal parts and hence always used two samples H_1 and H_2 in that region. Their values have been choosen by applying the optimisation technique such that it minimises the maximum deviation in the stop band. Future work may be suggested for defining the transition region more efficiently. Figure (4.5b) shows a designed characteristic of such a filter with order $N = 14$. As can be seen from the above mentioned two characteristics by using two transition samples and increasing N, the stop band attenuation is increased tremendously.

In many design problem it is usual practice to specify the pass band frequency F_p stop band frequency F_s tolarable deviations δ_1 and δ_2 in pass band and stopband respectively, Now, the designer's duty is to find out the required order N of the filter and the equation that optimum filter satisfying the specifications. To solve such problems we have proceeded as below:-

Based on measurements on an extensive set of optimal, linear phase, low-pass filters, Herrmann[9] emperically determined the relationship between δ_1, δ_2, F_p, F_s and ΔF.

as, $D = D_\infty(\delta_1, \delta_2) - f(\delta_1, \delta_2)\,(\Delta F)^2$ 4.3

where, $D = (N-1)\Delta F$ 4.4

and, $$D_{\infty}(\delta_1, \delta_2) = \left[a_1(\log_{10}\delta_1)^2 + a_2\log_{10}\delta_1 + a_3\right] \cdot \log_{10}\delta_2$$

$$+\left[a_4(\log_{10}\delta_1)^2 + a_5\log_{10}\delta_1 + a_6\right] \quad \ldots\ldots \qquad 4.5$$

with,

$$a_1 = 5.309 \times 10^{-3}$$
$$a_2 = 7.114 \times 10^{-2}$$
$$a_3 = -4.761 \times 10^{-1}$$
$$a_4 = -2.66 \times 10^{-3}$$
$$a_5 = -5.941 \times 10^{-1}$$
$$a_6 = -4.278 \times 10^{-1}$$

and

$$f(\delta_1, \delta_2) = b_1 + b_2\log_{10}\delta_1 - b_2\log_{10}\delta_2 \quad \ldots\ldots \qquad 4.6$$

with $b_1 = 11.01217$

$b_2 = 0.51244$

The coefficients in (4.5) and (4.6) were determined by a minimum mean - square error fitting procedure to the data, whereas the forms for (4.3) and (4.4) were suggested by some simple data fitting procedures.

For the design of the filter, the given specified parameters are F_p, F_s, δ_1 and δ_2. The unspecified parameter is N. In this case Equations (4.3) and 4.4) may be used to give N, the estimate of N, as

$$N = \frac{D_\infty(\delta_1, \delta_2)}{\Delta F} - f(\delta_1, \delta_2)\,(\Delta F)^2 + 1 \quad \ldots\ldots \qquad 4.7$$

Figure (4.7) shows the logic required to obtain the actual value of N that is required. After estimating $\hat{N}$ from (4.7), the direction parameter JD is initialized to 0. The parameter $\hat{N}$, Fp, F_s, δ_1 and δ_2 are used as input to the optimal design algorithm (Hooke and Jeeves pattern search technique) that returns the value $\hat{\delta}_2$ as the actual deviation in the stopband. This value is compared with δ_2 and if they are equal the algorithm goes for a second check about δ_1. If $\hat{\delta}_2 > \delta_2$ then $\hat{N}$ is incremented by 2(i.e. one filter order) and a check is made to see if the direction parameter JD was -1, indicating that $\hat{N}$ had previously been decreasing. If so, the new value of $\hat{N}$ is the smallest N that meets the requirement for δ_2 and the algorithm goes to point(2).If not, the value of JD is set to 1 and the updated value of $\hat{N}$ is used as input to the optimal design algorithm. A similar path is taken if $\hat{\delta}_2 < \delta_2$ whereby if JD was 1, the current value of $\hat{N}$ is the minimum value of N. Otherwise $\hat{N}$ is decreased by 2 and JD set to -1 and the algorithm repeats.

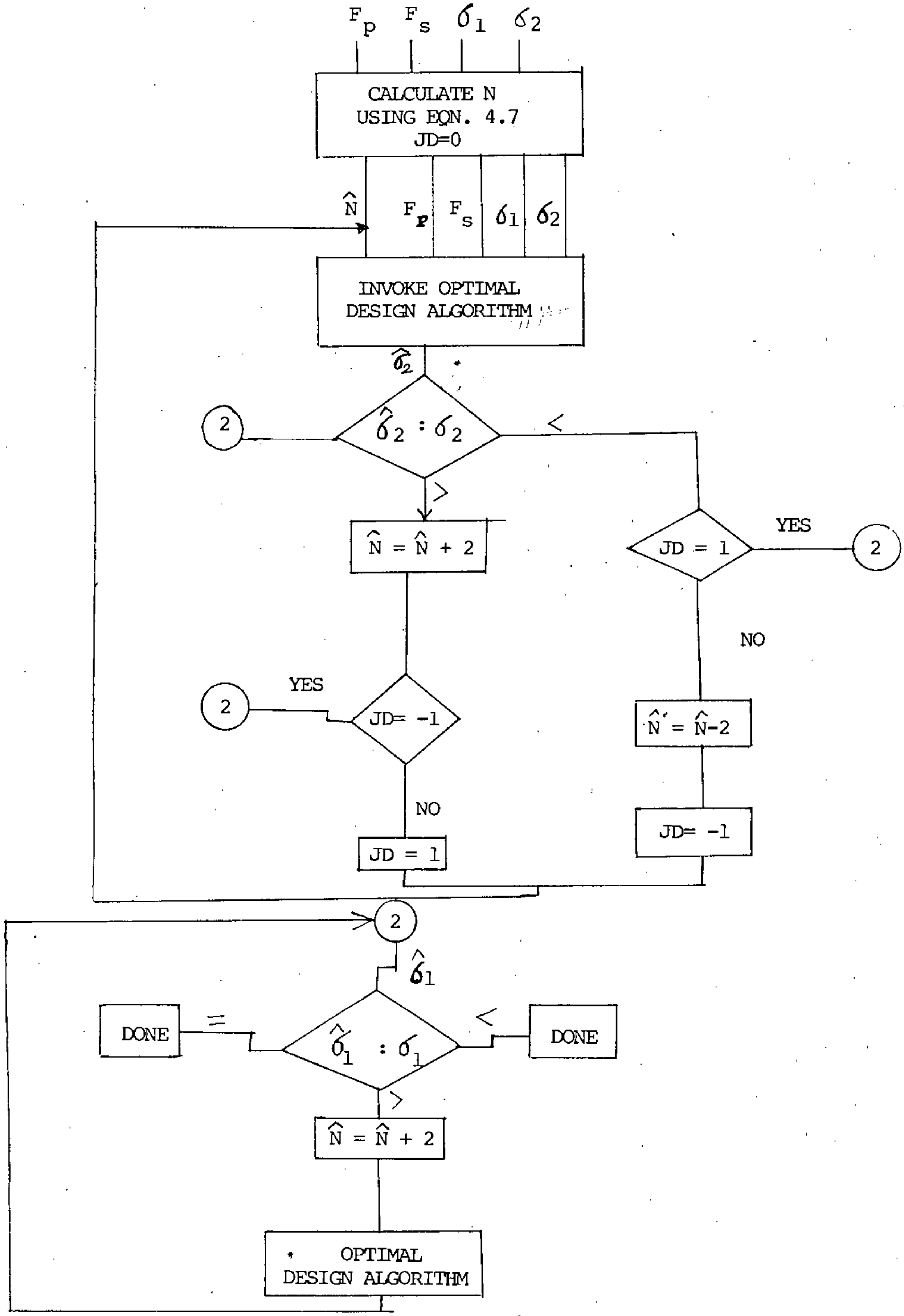


Fig 4.7. Algorithm for choosing smallest $\hat{N}$ to meet specifications on F_p, F_s, δ_1 and δ_2.

The process for attaining the value of δ_1 within the specified tolerance is also explained in the flow chart 4.7

Now for an example, we have designed an optimal low pass FIR filter with the following specifications,

$$\delta_1 = 0.0001$$

$$\delta_2 = 0.0003$$

$$F_p = 1.6$$

$$F_s = 1.8$$

The resulted frequence response of the transfer function is show in the fig. 4.8.

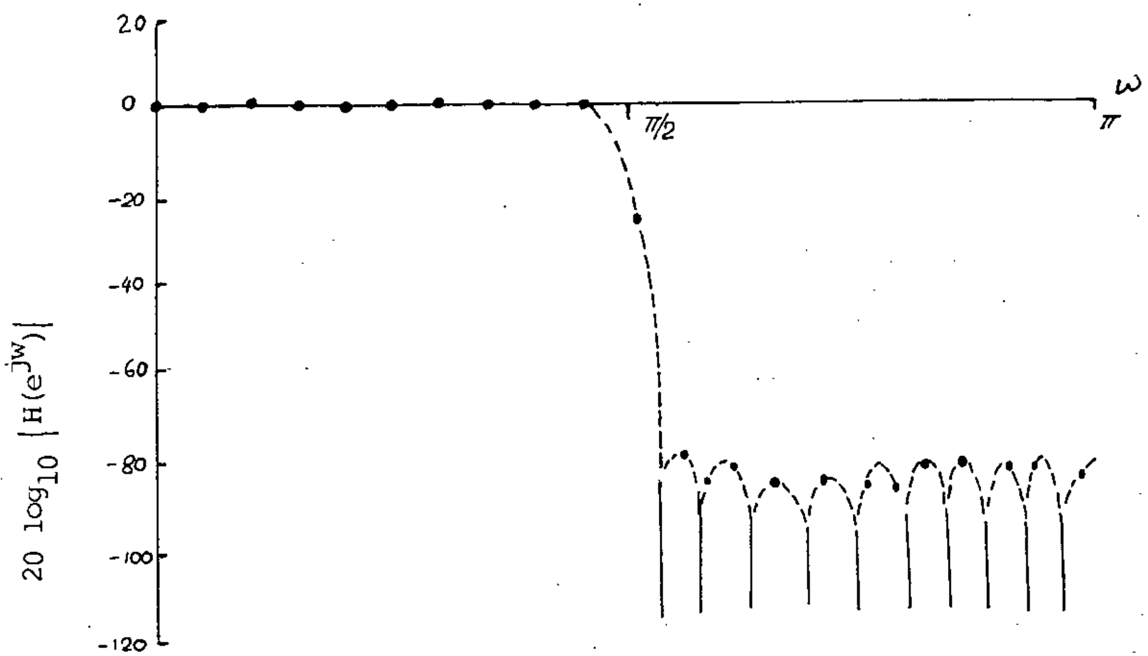


Fig. 4.8. Frequency response of the filter with given specifications (N = 26).

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

5.1 CONCLUSIONS:

In the present work, a design technique using Hooke and Jevees pattern search technique has been implemented. The design procedure is based on frequency sampling method. Optimization technique has also been implemented to define the transition region of the desired filter to maximise the stop band attenuation.

The present method for designing the low-pass FIR filter the results which we have obtained are quite satisfactory and the resulted filter could be rightly claimed to be close to the 'best' filter for that particular value of N (derived from the iterative loops of the algorithm shown in fig. 4.7).

5.2 SUGGESTIONS FOR FUTURE WORK:

In this method we have taken the Hooke and Jeeves pattern search optimization technique. For further analysis different optimization techniques can be used and a modification of the present algorithm can be made in a view to improve the flexibility of the total program.

In our work we have concentrated on the design of low-pass filter only. Future work can be extended for other FIR filters such as high-pass, band-pass and band-reject types. New approaches for defining the transition region points efficiently can be investigated.

REFERENCES

1. ANDREAS ANTONIOU, "Digital Filters Analysis and Design", McGraw - Hill, N.Y., 1979.

2. BERNARD GOLD & K.L. JORDAN, 'A Direct Search Procedure for Designing Finite Duration Impulse Response Filters", IEEE Trans, Audio Electroacoust, vol. AU-17, NO-1, March 1969, pp. 33-36.

3. CHENG, D.K., "Analysis of Linear Systems", Addision-Wesley, Massachusetts, 1977.

4. DAVID M. HIMMELBLAU, "Applied Nonlinear Programming", McGRAW-HILL, N.Y., 1972.

5. HARRY, Y-F. LAM., "Analog and Digital Filters, Prentice-Hall, N.J., 1979.

6. LAWRENCE R. RABINER, "Techniques for Designing Finite-Duration Impulse-Response Digital Filters", IEEE Trans, Communication Tech, VOL, COM-19, NO-2, April 1971, pp. 188-195.

7. LAWRENCE R. RABINER, "Linear Program Design of Finite Impulse Response (FIR) Digital Filters," IEEE Trans, Audio Electro-acoust, Vol. AU-20, No-4, OCT 1972, pp. 280-288

8. LAWRENCE R. RABINER & OTTO HERRMANN, "On the Design of Optimum FIR Low-Pass Filters with Even Impulse Response Duration", IEEE Trans, Audio Electroacoust, Vol. AU-21, No-4, Aug 1973, pp. 329-336.

9. LAWRENCE R. RABINER & OTTO HERRMANN, "The Predictability of Certain Optimum Finite Impulse Response Digital Filters", IEEE trans, Circuit Theory, Vol, CT-20, No-4, July 1973, pp. 401-408.

10. RABINER, L.R., and B. GOLD, "Theory and Application of Digital Signal Processing", Prentice-Hall, N.J., 1975.

11. R.W. HAMING, "Digital Filters", Printice Hall, N.J., 1977.

12. STANLEY, WILLIAM D., 'Digital Signal Processing', Reston Publishing Company, INC., VA., 1975.

13. THOMAS, W. PARKS & JAMES, H. McCLELLAN, "A Program for the Design of Linear Phase Finite Impulse Response Digital Filters", IEEE Trans, Audio Electroacoust, Vol. AU-20, No-3, Aug 1972, pp. 195-199.

14. LAWRENCE R. RABINER, "Approximate Design Relationships for how Pass FIR Digital Filters", IEEE Trans, Audio & Electroacoust, Vol. AU-21, No. 5, OCT. 1973, pp. 456-460.

APPENDIX - A

THE Z TRANSFORMATION

Although the conventional Laplace transformation can be used, the analysis of discrete-time signals (sampled-data systems) is greatly facilitated by the introduction of the z transformation, especially when responses only at the sampling instants are desired.

A sampler is needed to generate the discrete-time signal of a continuous input signal. The function of a sampler is illustrated in fig. 1.2, where the sampler is represented by a simple switch. If the duration of the sampling pulses is small in comparison with the time constants of the system of which the sampler is a part, the sampled output f(nT) or f(n) can be considered as a sequence of impulses occuring at the sampling instants, 0,T,2T,3T, ..., the strengths of the individual impulses being equal to the values of the input function f(t) at the respective instants.

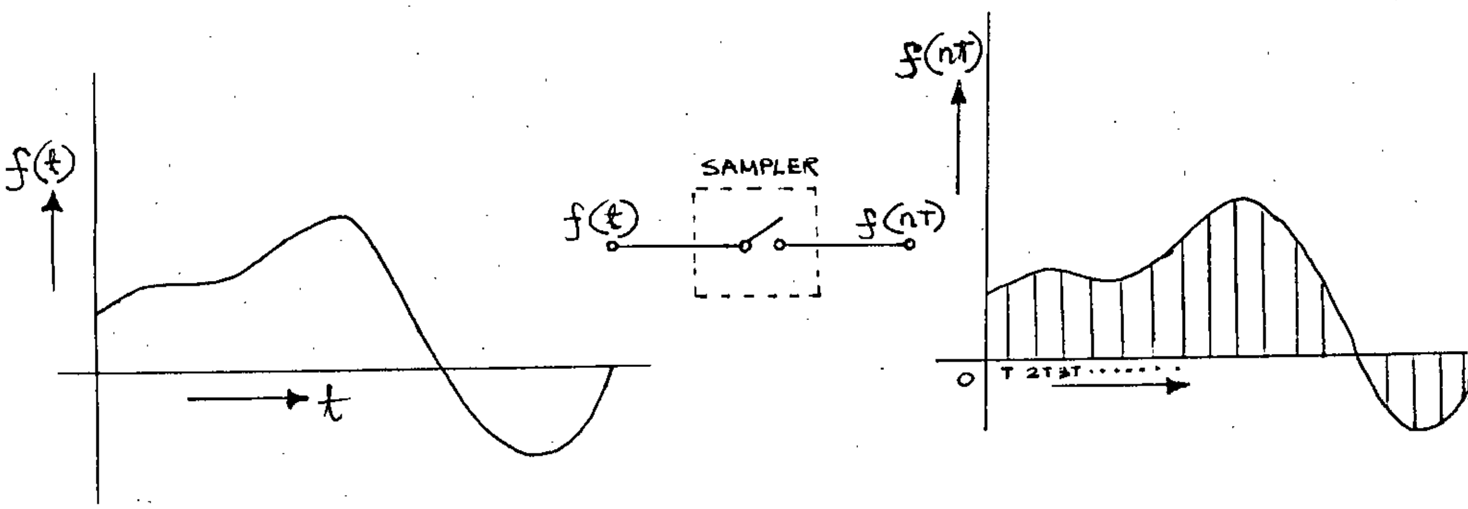


Fig.1.1. Function of a sampler

Thus we can write[1],

$$f(nT) = f(t)\, \delta_T(t),$$

where, f(nT) is the discrete, sampled function, and $\delta_T(t)$ represents a periodic train of unit impulses spaced T seconds apart:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT), \ \ldots\ldots \qquad 1.1$$

when, T is the sampling rate.

If f(t) = 0 for $t < 0$, Eq. (1-1) becomes, with the help of convolution summation

$$f(nT) = \sum_{n=0}^{\infty} f(nT)\, \delta(t-nT) \ \ldots \qquad 1.2$$

$$\text{Or } f(n) = \sum_{n=0}^{\infty} f(nT)\, \delta(t-nT) \ \ldots \qquad 1.3$$

as $f(nT) \equiv f(n)$.

Now taking the Laplace transform of the sampled output function f(n) and denoting the result by F(S); we have

$$F(s) = L\left[f(n)\right] = L\left[\sum_{n=o}^{\infty} f(nT)\, \delta(t-nT)\right]$$

$$= \sum_{n=o}^{\infty} f(nT)\, e^{-nTs} \ \ldots\ldots\ldots\ldots \qquad 1.4$$

since, $L\left[k\,\delta(t)\right] = k$

or $L\left[k\,\delta(t-t_o)\right] = k\,e^{-toS}$

Since s appears in Eq. 1-4, only in the exponential factor, it is convenient to introduce a new symbol:

$$z = e^{Ts} \quad \ldots\ldots\ldots \qquad 1.5$$

we have from Eqs, (1.4) & (1.5)

$$F(z) = \sum_{n=o}^{\infty} f(nT)\,z^{-n} = z\left[f(n)\right] \quad \ldots\ldots \qquad 1.6$$

Equation (1.6) defines the z transformation, F(z), of the sampled function f(nT).

Hence the z - transform x(z) of a sequence x(n) is defined to be

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n)\ z^{-n} \quad \ldots\ldots \qquad 1.7$$

where, z is a complex variable. Hence, X(z) is complex quantity.

The z - transform method is a very useful tool in solving linear difference equations. It reduces the solutions of such equations into those of algebraic equations. The application of z - transforms to a set of difference equations is analogous to the application of Laplace transforms to a set of differential equations.

1.2 APPLICATION OF Z - TRANSFORMATION TO OPEN-LOOP SYSTEMS

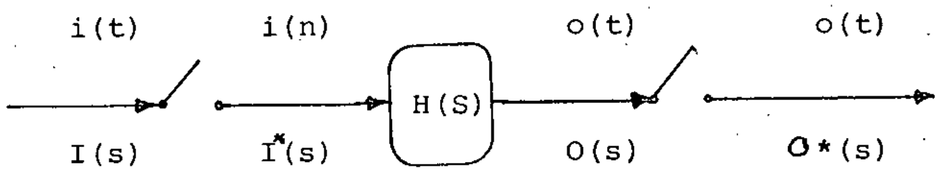


Fig. 1.3. A system with sampled input function.

The block-diagram representation of a simple sampled - data system in which the input function i(t) is periodically sampled by the sampler is shown in Fig. 1.3. The transfer function, H(S), is the ratio of the Laplace transform of the output function to the Laplace. transform of the input function, irrespective of whether the input function is continuous or sampled. Thus, for the situation in the above fig. can be expressed as,

$$O(S) = H(S) I^*(S)$$

where, $I^*(S)$ is the Laplace transform of the sampled input function.

It is analogous to the Laplace transform that the z - transformed relationship for a system in which both the input and the output are sampled in synchronism is simply,

$$O(z) = H(z)\ I\ (z)\ \ldots\ldots \qquad 1.8$$

where, $H(z) = z[h(nT)]$ i.e. z transform of the sampled Impulse response.

$I(z)$ = z transform of the sampled input

& $O(z)$ = z transform of the sampled output signal.

1.3 MAPPING OF JW-AXIS IN S PLANE ONTO Z PLANE AND ITS SIGNIFICANCE IN STABILITY CRITERION OF SAMPLED-DATA FEEDBACK SYSTEMS

We have, $z = e^{ST}$ 1.9

But we have, $S = \sigma + jw$ 1.10

using Eq. 1.10, in 1.9, we have,

$z = e^{\sigma T} \; e^{jwT}$ 1.11

where T is the sampling period.

On the imaginary axis of the s-plane,

$\sigma = 0$, Hence, $z\Big|_{\sigma=0} = e^{jwT}$ 1.12

$$|z|\Big|_{\sigma=0} = 1$$

Equation (1.12) shows very clearly that the jw-axis of the s-plane maps onto the z-plane as a unit circle centred at the origin. As the angular frequency w is increased from $-\infty$ through o to $+\infty$, the unit circle is traced over and over again every time wT goes through an angle of 2π radians.

For points in the right half of the s-plane, $\sigma > 0$ and from Eq. 1.11, $|z| > 1$. Thus, the right half of the s-plane maps onto the z-plane as the region exterior to the unit circle, correspondingly, the left half of the s-plane maps

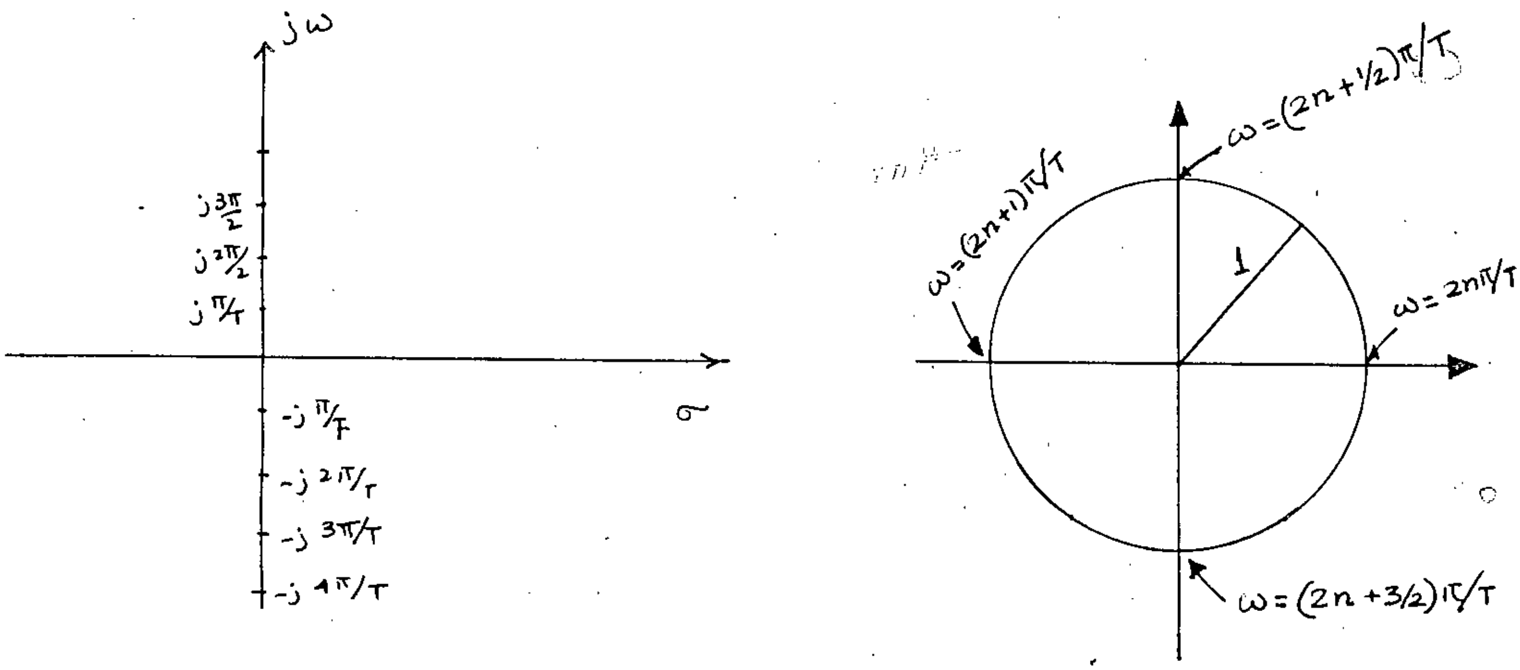


Fig. 1.4. Mapping of jw-axis in s-plane onto z-plane.

a) jw-axis in s-plane.

b) Unit circle in z-plane (n=any integer from $-\infty$ to $+\infty$)

over as the interior of the unit circle.

The necessary and sufficient condition for a sampled - data feedback system to be stable is that all the poles of its over-all transfer function lie inside the unit circle in the z-plane. An alternative statement for the stability requirement is: the necessary and sufficient condition for a sampled - data feedback system to be stable is that all the roots of its characteristics equation in z, have an absolute value less than one.

APPENDIX B

Z-Transform Pairs of Some Causal Sequences

	Causal Sequence x(n)	Z-Transforms of Causal Sequences
	$x(n)=0$ for $n<0$	$X(z) = \sum_{n=o}^{\infty} x(n)z^{-n}$
1	$x(n) = \delta(n)$	$X(z) = 1$
2	$x(n) = \delta(n-m)$	$X(z) = z^{-m}$
3	$x(n) = u(n)$	$X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$
4	$x(n) = a^n u(n)$	$X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$
5	$x(n) = nu(n)$	$X(z) = \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$
6	$x(n) = [a^n \sin nwT]\, u(n)$	$X(z) = \frac{az \sin wT}{z^2-2az \cos wT+a^2}$ $= \frac{az^{-1} \sin wT}{a^2z^{-2}-2az^{-1}\cos wT+1}$
7	$x(n) = [a^n \cos nwT]\, u(n)$	$X(z) = \frac{z(z-a \cos wT)}{z^2-2az \cos wT +a^2}$ $= \frac{1-az^{-1} \cos wT}{a^2z^{-2}-2az^{-1} \cos wT+1}$